

Chapter 2

THE FIBONACCI AND LUCAS NUMBERS

The great Italian mathematician, Leonardo of Pisa (c. 1170-1250), who is known today as Fibonacci (an abbreviation of filius Bonacci), expanded on the Arabic algebra of North Africa and introduced algebra into Europe. The solution of a problem in his book *Liber Abacci* uses the sequence

(F) $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ddot{y}$

One of the many applications of this **Fibonacci sequence** is a theorem about the number of steps in an algorithm for finding the greatest common divisor of a pair of large integers. We study the sequence here because it provides a wonderful opportunity for discovering mathematical patterns.

The numbers shown in (F) are just the beginning of the unending Fibonacci sequence. The rule for obtaining more terms is as follows:

RECURSIVE PROPERTY. The sum of two consecutive terms in (F) is the term immediately after them.

For example, the term after 55 in (F) is $34 + 55 = 89$ and the term after that is $55 + 89 = 144$.

To aid in stating properties of the Fibonacci sequence, we use the customary notation $F_0, F_1, F_2, F_3, \ddot{y}$ for the integers of the Fibonacci sequence. That is, $F_0 = 0, F_1 = 1, F_2 = F_0 + F_1 = 1, F_3 = F_1 + F_2 = 2, F_4 = F_2 + F_3 = 3, F_5 = F_3 + F_4 = 5, F_6 = F_4 + F_5 = 8$, and so on. When F_n stands for some term of the sequence, the term just after F_n is represented by F_{n+1} , the term after F_{n+1} is F_{n+2} , and so on. Also, the term just before F_n is F_{n-1} , the term just before F_{n-1} is F_{n-2} and so on.

We now can define the Fibonacci numbers formally as the sequence F_0, F_1, \ddot{y} having the two following properties.

INITIAL CONDITIONS $F_0 = 0$ and $F_1 = 1$.

RECURSION RULE $F_n + F_{n+1} = F_{n+2}$ for $n = 0, 1, 2, \ddot{y}$.

Next let S_n stand for the sum of the Fibonacci numbers from F_0 through F_n . That is,

$$\begin{aligned} S_0 &= F_0 = 0 \\ S_1 &= F_0 + F_1 = 0 + 1 = 1 \\ S_2 &= F_0 + F_1 + F_2 = 0 + 1 + 1 = 2 \\ S_3 &= F_0 + F_1 + F_2 + F_3 = S_2 + F_3 = 2 + 2 = 4 \end{aligned}$$

and in general, $S_n = F_0 + F_1 + F_2 + \ddot{y} + F_n = S_{n-1} + F_n$.

We tabulate some of the values and look for a pattern.

n	0	1	2	3	4	5	6	7	\ddot{y}
F_n	0	1	1	2	3	5	8	13	\ddot{y}
S_n	0	1	2	4	7	12	20	33	\ddot{y}

Is there a relationship between the numbers on the third line of this table and the Fibonacci numbers? One pattern is that each of the terms of the sequence S_0, S_1, S_2, \ddot{y} is 1 less than a Fibonacci number. Specifically, we have

$$\begin{aligned}
S_0 &= F_2 - 1 = 0 \\
S_1 &= F_3 - 1 = 1 \\
S_2 &= F_4 - 1 = 2 \\
S_3 &= F_5 - 1 = 4 \\
S_4 &= F_6 - 1 = 7 \\
S_5 &= F_7 - 1 = 12
\end{aligned}$$

and might conjecture that $S_n = F_{n+2} - 1$ for $n = 0, 1, 2, \ddot{y}$. Does this formula hold for $n = 6$? Yes, since

$$\begin{aligned}
S_6 &= F_0 \% F_1 \% F_2 \% F_3 \% F_4 \% F_5 \% F_6 \\
&= S_5 \% F_6 \\
&= (F_7 \& 1) \% F_6 \\
&= (F_6 \% F_7) \& 1 \\
&= F_8 \& 1.
\end{aligned}$$

The first steps in proving our conjecture correct for all the terms in the unending sequence S_0, S_1, S_2, \ddot{y} are rewriting the recursion formula $F_n + F_{n+1} = F_{n+2}$ as $F_n = F_{n+2} - F_{n+1}$ and then using this to replace each Fibonacci number in the sum S_n by a difference as follows:

$$\begin{aligned}
S_n &= F_0 \% F_1 \% F_2 \% \ddot{y} \% F_{n+1} \% F_n \\
S_n &= (F_2 \& F_1) \% (F_3 \& F_2) \% (F_4 \& F_3) \% \ddot{y} \% (F_{n+1} \& F_n) \% (F_{n+2} \& F_{n+1}).
\end{aligned}$$

Next we rearrange the terms and get

$$\begin{aligned}
S_n &= \& F_1 \% (F_2 \& F_2) \% (F_3 \& F_3) \% \ddot{y} \% (F_{n+1} \& F_{n+1}) \% F_{n+2} \\
S_n &= \& F_1 \% 0 \% 0 \% \ddot{y} \% 0 \% F_{n+2} \\
S_n &= F_{n+2} \& F_1 = F_{n+2} \& 1.
\end{aligned}$$

Thus we have made our conjecture (that is, educated guess) into a theorem.

The fundamental relation $F_n = F_{n+2} - F_{n+1}$ can also be used to define F_n when n is a negative integer. Letting $n = -1$ in this formula gives us $F_{-1} = F_1 - F_0 = 1 - 0 = 1$. Similarly, one finds that $F_{-2} = F_0 - F_{-1} = 0 - 1 = -1$ and $F_{-3} = F_{-1} - F_{-2} = 1 - (-1) = 2$. In this way one can obtain F_n for any negative integer n .

Some of the values of F_n for negative integers n are shown in the following table:

n	...	-6	-5	-4	-3	-2	-1
F_n	...	-8	5	-3	2	-1	1

Perhaps the greatest investigator of properties of the Fibonacci and related sequences was François Edouard Anatole Lucas (1842-1891). A sequence related to the F_n bears his name. The **Lucas sequence**, 2, 1, 3, 4, 7, 11, 18, 29, 47, ..., is defined by

$$L_0 = 2, L_1 = 1, L_2 = L_1 + L_0, L_3 = L_2 + L_1, \dots, L_{n+2} = L_{n+1} + L_n, \dots$$

Some of the many relations involving the F_n and the L_n are suggested in the problems below. These are only a very small fraction of the large number of known properties of the Fibonacci and Lucas numbers. In fact, there is a mathematical journal, *The Fibonacci Quarterly*, devoted to them and to related material.

Problems for Chapter 2

1. For the Fibonacci numbers F_n show that:

$$(a) F_3 = 2F_1 + F_0. \quad (b) F_4 = 2F_2 + F_1. \quad (c) F_5 = 2F_3 + F_2.$$

2. The relation $F_{n+2} = F_{n+1} + F_n$ holds for all integers n and hence so does $F_{n+3} = F_{n+2} + F_{n+1}$. Combine these two formulas to find an expression for F_{n+3} in terms of F_{n+1} and F_n .

3. Find r , given that $F_r = 2F_{101} + F_{100}$.

4. Express $F_{157} + 2F_{158}$ in the form F_s .

5. Show the following:

$$(a) F_4 = 3F_1 + 2F_0. \quad (b) F_5 = 3F_2 + 2F_1.$$

6. Add corresponding sides of the formulas of the previous problem and use this to show that $F_6 = 3F_3 + 2F_2$.

7. Express F_{n+4} in terms of F_{n+1} and F_n .

8. Find s , given that $F_s = 3F_{200} + 2F_{199}$.

9. Find t , given that $F_t = 5F_{317} + 3F_{316}$.

10. Find numbers a and b such that $F_{n+6} = aF_{n+1} + bF_n$ for all integers n .

11. Show the following:

$$\begin{array}{ll} \text{(a)} F_0 + F_2 + F_4 + F_6 = F_7 - 1. & \text{(b)} F_0 + F_2 + F_4 + F_6 + F_8 = F_9 - 1. \\ \text{(c)} F_1 + F_3 + F_5 + F_7 = F_8. & \text{(d)} F_1 + F_3 + F_5 + F_7 + F_9 = F_{10}. \end{array}$$

12. The relation $F_{n+2} = F_{n+1} + F_n$ can be rewritten as $F_{n+1} = F_{n+2} - F_n$. Use this form to find a compact expression for $F_a + F_{a+2} + F_{a+4} + F_{a+6} + \dots + F_{a+2m}$.

13. Find p , given that $F_p = F_1 + F_3 + F_5 + F_7 + \dots + F_{701}$.

14. Find u and v , given that $F_u - F_v = F_{200} + F_{202} + F_{204} + \dots + F_{800}$.

15. Show the following:

$$\text{(a)} F_4 = 3F_2 - F_0. \quad \text{(b)} F_5 = 3F_3 - F_1. \quad \text{(c)} F_6 = 3F_4 - F_2.$$

16. Use the formulas $F_{n+4} = 3F_{n+1} + 2F_n$ and $F_{n+1} = F_{n+2} - F_n$ to express F_{n+4} in terms of F_{n+2} and F_n .

17. Show the following:

$$\text{(a)} 2(F_0 + F_3 + F_6 + F_9 + F_{12}) = F_{14} - 1. \quad \text{(b)} 2(F_0 + F_3 + F_6 + F_9 + F_{12} + F_{15}) = F_{17} - 1.$$

18. Show the following:

$$\text{(a)} 2(F_1 + F_4 + F_7 + F_{10} + F_{13}) = F_{15}. \quad \text{(b)} 2(F_1 + F_4 + F_7 + F_{10} + F_{13} + F_{16}) = F_{18}.$$

19. By addition of corresponding sides of formulas of the two previous problems, find expressions for:

$$\text{(a)} 2(F_2 + F_5 + F_8 + F_{11} + F_{14}). \quad \text{(b)} 2(F_2 + F_5 + F_8 + F_{11} + F_{14} + F_{17}).$$

20. (i) Prove that $F_{n+3} - 4F_n - F_{n-3} = 0$ for $n \geq 3$.

(ii) Prove that $F_{n+4} - 7F_n + F_{n-4} = 0$ for $n \geq 4$.

*(iii) For $m \geq 4$, find a compact expression for $F_a + F_{a+4} + F_{a+8} + \dots + F_{a+4m}$.

21. Evaluate each of the following sums:

$$\begin{array}{ll} \text{(a)} \binom{2}{0} \% \binom{1}{1}. & \text{(b)} \binom{3}{0} \% \binom{2}{1}. \\ \text{(c)} \binom{4}{0} \% \binom{3}{1} \% \binom{2}{2}. & \text{(d)} \binom{5}{0} \% \binom{4}{1} \% \binom{3}{2}. \\ \text{(e)} \binom{6}{0} \% \binom{5}{1} \% \binom{4}{2} \% \binom{3}{3}. & \text{(f)} \binom{7}{0} \% \binom{6}{1} \% \binom{5}{2} \% \binom{4}{3}. \end{array}$$

22. Find m , given that $\begin{pmatrix} 9 \\ 0 \end{pmatrix} \% \begin{pmatrix} 8 \\ 1 \end{pmatrix} \% \begin{pmatrix} 7 \\ 2 \end{pmatrix} \% \begin{pmatrix} 6 \\ 3 \end{pmatrix} \% \begin{pmatrix} 5 \\ 4 \end{pmatrix} = F_m$.

23. Find r , s , and t given that:

(a) $\begin{pmatrix} 2 \\ 0 \end{pmatrix}_{F_0} \% \begin{pmatrix} 2 \\ 1 \end{pmatrix}_{F_1} \% \begin{pmatrix} 2 \\ 2 \end{pmatrix}_{F_2} = F_r$. (b) $\begin{pmatrix} 2 \\ 0 \end{pmatrix}_{F_1} \% \begin{pmatrix} 2 \\ 1 \end{pmatrix}_{F_2} \% \begin{pmatrix} 2 \\ 2 \end{pmatrix}_{F_3} = F_s$.

(c) $\begin{pmatrix} 2 \\ 0 \end{pmatrix}_{F_7} \% \begin{pmatrix} 2 \\ 1 \end{pmatrix}_{F_8} \% \begin{pmatrix} 2 \\ 2 \end{pmatrix}_{F_9} = F_t$.

24. Find u , v , and w , given that:

(a) $\begin{pmatrix} 3 \\ 0 \end{pmatrix}_{F_0} \% \begin{pmatrix} 3 \\ 1 \end{pmatrix}_{F_1} \% \begin{pmatrix} 3 \\ 2 \end{pmatrix}_{F_2} \% \begin{pmatrix} 3 \\ 3 \end{pmatrix}_{F_3} = F_u$.

(b) $\begin{pmatrix} 3 \\ 0 \end{pmatrix}_{F_1} \% \begin{pmatrix} 3 \\ 1 \end{pmatrix}_{F_2} \% \begin{pmatrix} 3 \\ 2 \end{pmatrix}_{F_3} \% \begin{pmatrix} 3 \\ 3 \end{pmatrix}_{F_4} = F_v$.

(c) $\begin{pmatrix} 3 \\ 0 \end{pmatrix}_{F_7} \% \begin{pmatrix} 3 \\ 1 \end{pmatrix}_{F_8} \% \begin{pmatrix} 3 \\ 2 \end{pmatrix}_{F_9} \% \begin{pmatrix} 3 \\ 3 \end{pmatrix}_{F_{10}} = F_w$.

25. Find r , s , and t , given that:

(a) $(F_7)^2 \% (F_8)^2 = F_r$. (b) $(F_8)^2 \% (F_9)^2 = F_s$. (c) $(F_9)^2 \% (F_{10})^2 = F_t$.

26. Find u , v , and w , given that:

(a) $(F_3)^2 \& (F_2)^2 = F_u F_{u\%3}$. (b) $(F_4)^2 \& (F_3)^2 = F_v F_{v\%3}$. (c) $(F_9)^2 \& (F_8)^2 = F_w F_{w\%3}$.

27. Let L_0, L_1, L_2, \dots be the Lucas sequence. Prove that

$$L_0 \% L_1 \% L_2 \% L_3 \% \dots \% L_n = L_{n\%2} \& 1.$$

28. Find r , given that $L_r = 2L_{100} \% L_{99}$.

29. Find s , given that $L_s = 3L_{201} \% 2L_{200}$.

30. Find t , given that $L_t = 8L_{999} \% 5L_{998}$.

31. Show that $L_0 \% L_2 \% L_4 \% L_6 \% L_8 \% L_{10} = L_{11} \% 1$.

32. Find m , given that $L_0 \% L_2 \% L_4 \% L_6 \% \dots \% L_{400} = L_m \% 1$.

33. Derive a formula for $L_1 \% L_3 \% L_5 \% L_7 \% \dots \% L_{2m\%1}$.
34. Conjecture, and test in several cases, formulas for:
- (a) $L_0 \% L_3 \% L_6 \% L_9 \% \dots \% L_{3m}$. (b) $L_1 \% L_4 \% L_7 \% L_{10} \% \dots \% L_{3m\%1}$.
- (c) $L_2 \% L_5 \% L_8 \% L_{11} \% \dots \% L_{3m\%2}$. (d) $\binom{n}{0} L_k \% \binom{n}{1} L_{k\%1} \% \binom{n}{2} L_{k\%2} \% \dots \% \binom{n}{n} L_{k\%n}$.
35. Find r , s , and t , given that $F_2 L_2 = F_r$, $F_3 L_3 = F_s$, and $F_4 L_4 = F_t$.
36. Find u , v , and w , given that $F_{10}/F_5 = L_u$, $F_{12}/F_6 = L_v$, and $F_{14}/F_7 = L_w$.
37. Evaluate the following:
- (a) $(F_1)^2$ & $F_0 F_2$. (b) $(F_2)^2$ & $F_1 F_3$. (c) $(F_3)^2$ & $F_2 F_4$. (d) $(F_4)^2$ & $F_3 F_5$.
38. Evaluate the expressions of the previous problem with each Fibonacci number replaced by the corresponding Lucas number.
39. Which of the Fibonacci numbers F_{800} , F_{801} , F_{802} , F_{803} , F_{804} , and F_{805} are even?
40. Which of the Fibonacci numbers of the previous problem are exactly divisible by 3?